Math6261 23-03-31

Review:

Let $X_1, X_2, \cdots$, be i.i.d. r.v's and set

$$S_n = X_1 + \cdots + X_n, \quad n = 1, 2, \cdots$$

The process (S_n) is called a random walk.

Thm 4.8: For a random walk on $\mathbb{R}$, there are only 4 possible cases, one of which has probability 1.

(i) $S_n = 0$ for all n.

(ii) $S_n \to +\infty$.

(iii) $S_n \to -\infty$

(iv) $-\infty = \liminf S_n < \limsup S_n = \infty$.

- Below we will further study the asymptotical behavior of S_n under the assumption that $E|X| < \infty$.

By the strong law of large numbers, if $\mu := EX \neq 0$, then

$$\lim_{n\to\infty} \frac{1}{n} S_n = \mu \quad \text{a.s.}$$

Hence

$$\lim S_n = \begin{cases} \infty & \text{if } \mu > 0 \\ -\infty & \text{if } \mu < 0. \end{cases}$$

It remains to consider the case when $\mu = EX = 0$.

Thm 4.9. (K.L. Chung & Fuchs)

Suppose that $EX = 0$. Then there are only 2 possible cases, one of which has probability 1.

(i) $S_n = 0$ for all n.

(ii) $-\infty = \liminf S_n < \limsup S_n = \infty$.

To prove the above result, let us have some preparation.

Recall that we can rebuild our prob. space as

$$(\Omega, \mathcal{F}, P) = \prod_{n=1}^{\infty} \left(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu \right),$$

where μ is the prob. law of X_1.

- Then we can define $X_n : \Omega \to \mathbb{R}$ as

$$X_n(\omega) = \omega_n \qquad \text{for} \qquad \omega = (\omega_1, \omega_2, \cdots) \in \Omega.$$

- Now we define $\sigma : \Omega \to \Omega$ by

$$\sigma\omega = (\omega_2, \omega_3, \omega_4, \cdots) \qquad \text{for} \qquad \omega = (\omega_1, \omega_2, \cdots).$$

We call σ the left shift map on Ω.

Lem 4.10. We have $P = P \circ \sigma^{-1}$. That is, for every $A \in \mathcal{F}$,

$$P(A) = P(\sigma^{-1}A).$$

pf. Set

$$\mathcal{T} = \{ A \in \mathcal{F} : \quad P(A) = P(\sigma^{-1}A) \}.$$

Then one can check that $\mathcal{T}$ is a λ-system, i.e. the following 3 properties hold:

① $\Omega \in \mathcal{T}$.
② If $A, B \in \mathcal{T}$ with $A \subset B$, then $B \setminus A \in \mathcal{T}$.
③ If $A_n \in \mathcal{T}$ and $A_n \uparrow A$, then $A \in \mathcal{T}$.

Next let $\mathcal{P}$ be the collection of rectangles in Ω; recall a set $C \in \mathcal{F}$ is called a rectangle if

$$C = \{\omega : \omega_1 \in C_1, \omega_2 \in C_2, \cdots, \omega_k \in C_k\}$$

for some $k \in \mathbb{N}$ and $C_1, \cdots, C_k \subset \mathcal{B}(\mathbb{R})$.

By definition, $\mathcal{F}$ is the sigma-algebra generated by $\mathcal{P}$.

Clearly $\mathcal{P}$ is a π-system, that is, $\mathcal{P}$ is closed under intersection.

Next we show that $\mathcal{P} \subset \mathcal{T}$.

Let $C = C_1 \times \cdots \times C_k \times \mathbb{R}^{\mathbb{N}} \subset \Omega$, with $C_i \in \mathcal{B}(\mathbb{R})$.
Then $\sigma^{-1} C = \mathbb{R} \times C_1 \times \cdots \times C_k \times \mathbb{R}^{\mathbb{N}}$.

clearly

$$P(C) = \mu(C_1) \times \cdots \times \mu(C_k) = P(\sigma^{-1} C).$$

So we have that $C \in \mathcal{T}$. Hence $\mathcal{P} \subset \mathcal{T}$.

By Dynkin's π-λ Theorem, $\mathcal{F} = \sigma(\mathcal{P}) \subset \mathcal{T}$.

Since $\mathcal{T} \subset \mathcal{F}$, it follows that $\mathcal{T} = \mathcal{F}$, i.e., $P = P \circ \sigma^{-1}$. $\blacksquare$

Lem 4.11. Let (a_n) be a sequence of real number such that

$$\lim_{n\to\infty} a_n = \infty.$$

Then there exist $\varepsilon > 0$ and $p \in \mathbb{N}$ such that

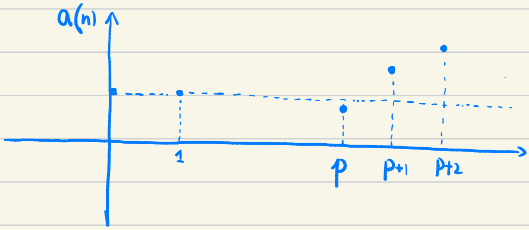
$$a_{n+p} - a_p > \varepsilon \quad \text{for all } n \geq 1.$$

Pf. Define

$$p = \sup\{k \geq 1 : a_k \leq a_1\}.$$

Then $p < \infty$ since $a_n \to \infty$ as $n \to \infty$.

Moreover $a_p \leq a_1$, and $a_{n+p} > a_1$ for all $n \geq 1$.



Since $a_n \to \infty$ as $n \to \infty$, $\exists$ n_0 s.t

$$a_{n+p} > a_p + 1 \quad \text{for all } n \geq n_0.$$

Take $\delta = \min\{a_{k+p} - a_p : k = 1, 2, \cdots, n_0\} > 0.$

Clearly,

$$a_{n+p} - a_p \geq \min\{\delta, 1\} \quad \text{for all } n \geq 1.$$

$\blacksquare$

Below we prove Thm 4.9 by following the argument of Dekking (1982).

Pf of Thm 4.9: By Thm 4.8, we only need to show that

$$P\left(\lim S_n = +\infty \right) = 0 \qquad (1)$$

$$P\left(\lim S_n = -\infty \right) = 0.$$

WLOG, we only prove (1).

For $\varepsilon > 0$ and $t \in \mathbb{R}$, write

$$I_\varepsilon(t) = [t-\varepsilon, t+\varepsilon].$$

For $n \in \mathbb{N}$ and $\omega \in \Omega$, define

$$R_n^\varepsilon(\omega) = \lambda\left(\bigcup_{i=1}^{n} I_\varepsilon\left(S_i(\omega)\right)\right).$$

Since $EX=0$, by the strong law of large numbers,

$$\lim_{n\to\infty} \frac{S_n(\omega)}{n} = 0 \quad \text{for a.s } \omega \in \Omega.$$

Hence for almost all ω and any $\delta > 0$, $\exists$ $n_0 = n_0(\omega, \delta)$ such that

$$|S_k(\omega)| \leq k\delta \quad \text{for all } k \geq n_0.$$

Therefore $I_\varepsilon(S_k(\omega)) \subset [-n\delta - \varepsilon, n\delta + \varepsilon]$ for $n_0 \leq k \leq n$.

It follows that

$$R_n^\varepsilon(\omega) \leq R_{n_0}^\varepsilon(\omega) + (2n\delta + 2\varepsilon) \quad \text{for all } n.$$

Hence

$$\overline{\lim_{n\to\infty}} \frac{R_n^\varepsilon(\omega)}{n} \leq \delta.$$

Therefore $\lim_{n\to\infty} \frac{R_n^\varepsilon(\omega)}{n} = 0$ a.s.

Notice that $\frac{R_n^\varepsilon(\omega)}{n} \leq 2\varepsilon$ for all $n \geq 0$, by the DCT,

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}(R_n^\varepsilon) = 0.$$

Observe that

$$S_n(\sigma\omega) = S_{n+1}(\omega) - S_1(\omega).$$

So

$$R_{n+1}^\varepsilon - R_n^\varepsilon \circ \sigma = \lambda\left(\bigcup_{i=1}^{n+1} I_\varepsilon(S_i)\right) - \lambda\left(\bigcup_{i=1}^{n} I_\varepsilon(S_{i+1} - S_1)\right)$$

$$= \lambda\left(\bigcup_{i=1}^{n+1} I_\varepsilon(S_i)\right) - \lambda\left(\bigcup_{i=2}^{n+2} I_\varepsilon(S_i)\right).$$

It follows that

$$R_{n+1}^{\varepsilon} - R_n^{\varepsilon} \circ \sigma \geqslant 2\varepsilon \, \mathbb{1}_{\{ |S_{i+1} - S_1| > 2\varepsilon \text{ for all } 1 \le i \le n \}}$$

Hence

$$\mathbb{E}(R_{n+1}^{\varepsilon}) - \mathbb{E}(R_n^{\varepsilon} \circ \sigma) \geqslant 2\varepsilon \cdot P(|S_{i+1} - S_1| > 2\varepsilon \text{ for all } 1 \le i \le n)$$

$$\geqslant 2\varepsilon \, P(|S_{i+1} - S_1| > 2\varepsilon \text{ for all } i \geq 1)$$

$$= 2\varepsilon \, P(|S_i| > 2\varepsilon \text{ for all } i \geq 1)$$

(Since the process $\{ S_{i+1} - S_1 : i \geq 1 \}$ and $\{ S_i : i \geq 1 \}$ have the same distribution.

As $P = P \circ \sigma^{-1}$, $\mathbb{E}(R_n^{\varepsilon} \circ \sigma) = E(R_n^{\varepsilon})$. It follows that

$$\underline{\lim} \frac{E(R_n^{\varepsilon})}{n} \geqslant 2\varepsilon \, P(|S_i| > 2\varepsilon \text{ for all } i \geq 1)$$

Since $\lim_{n \to \infty} \frac{E(R_n^{\varepsilon})}{n} = 0$, so

$$P(|S_i| > \varepsilon \text{ for all } i \geq 1) = 0 .$$

Again, Since $\{ S_{i+p} - S_p : i \geq 1 \}$ has the same distribution as $\{ S_i : i \geq 1 \}$,

it follows that for all $\varepsilon > 0$ and $p \in \mathbb{N}$,

$$P(| S_{i+p} - S_p | > \varepsilon \text{ for all } i \geq 1) = 0 .$$

Hence

$$F := \bigcup_{\varepsilon\in\mathbb{Q}_+} \bigcup_{p\in\mathbb{N}} \left\{ \omega : \left| S_{i+p}^{(\omega)} - S_{p(\omega)} \right| > \varepsilon \text{ for all } i \geq 1 \right\}$$

has prob. 0.

However by Lem 4.9

$$\left\{ \omega : \lim_{n\to\infty} S_n(\omega) = +\infty \right\} \subset F.$$

Hence

$$P\left(\lim S_n = +\infty \right) = 0. \qquad \blacksquare$$

In what follows, we extend the result of Chung and Fuchs to general ergodic stationary processes.

Def. Let $(\Omega, \mathcal{F}, P)$ be a prob. space and $T : \Omega \to \Omega$ a measurable mapping such that

$$P(A) = P(T^{-1}A) \quad \text{for every } A \in \mathcal{F}.$$

Say that T is ergodic with respect to P if for each $A \in \mathcal{F}$ with $A = T^{-1}A$, one has either $P(A) = 0$

or $P(A)=1$.

The following ergodic Thm generalizes the SLLN:

<u>Thm 4.12</u>. (Birkhoff ergodic Thm)

Let $T: \Omega \to \Omega$ be an ergodic measure preserving transformation on a probability space $(\Omega, \mathcal{F}, P)$. Let $f \in L^1(\Omega, \mathcal{F}, P)$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\omega) = \int f \, dP \quad \text{a.s.}$$

Moreover, the following result generalizes Chung-Fuchs' theorem:

Thm 4.13 (Dekking, 1982)

Under the same assumptions of Thm 4.13, suppose that

$$P\left\{\omega : \lim_{n\to\infty} \sum_{k=0}^{n-1} f(T^k\omega) = +\infty\right\} > 0.$$

Then $\int f \, dP > 0$.